1 Online appendix (not for publication)

1.1 Appendix A: Proof of Proposition 2.1

Proof of Proposition 2.1. We first derive the implementability condition. Define the Arrow-Debreu price $q_t^0 \equiv \sum_{i=0}^{t-1} R_i^{-1}$ for $t \geq 1$, with the numeraire $q_0^0 = 1$. From the consumer's first order conditions $u_c(t) = \lambda_t$ and $\frac{\lambda_t}{R_t} = \beta\lambda_{t+1}$, we have

$$q_t^0 = \beta^t \frac{u_c(t)}{u_c(0)}. \tag{1}$$

Iterating the household's flow budget constraint from the time 0, we obtain the present-value budget constraint that

$$b_0 = \sum_{t=0}^{\infty} q_t^0 \left\{ c_t - (1-\tau_t^n) w_t n_t + \underbrace{k_{t+1} - \left[\left(1-\tau_t^k\right) r_t + 1 - \delta\right] k_t}_{\equiv x_t} \right\} + \lim_{T\to\infty} q_t^0 b_T. \tag{2}$$

The term $\sum_{t=0}^{\infty} q_t^0 x_t$ in (2) is derived as

$$\begin{aligned}
&\sum_{t=0}^{\infty} q_t^0 x_t \qquad (3)\\
&= \sum_{t=0}^{\infty} q_t^0 \left\{k_{t+1} - \left[\left(1-\tau_t^k\right) r_t + 1 - \delta\right] k_t\right\}\\
&= \lim_{T\to\infty} \sum_{t=0}^{T} q_t^0 \left\{k_{t+1} - \left[\left(1-\tau_t^k\right) r_t + 1 - \delta\right] k_t\right\}\\
&= \lim_{T\to\infty} \left\{ \sum_{t=0}^{T} q_t^0 k_{t+1} - \sum_{t=1}^{T} q_t^0 \left[\left(1-\tau_t^k\right) r_t + 1 - \delta\right] k_t \right\} - \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0\\
&= \lim_{T\to\infty} \sum_{t=0}^{T-1} \left\{q_t^0 - q_{t+1}^0 \left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right]\right\} k_{t+1} - \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + \lim_{T\to\infty} q_T^0 k_{T+1}\\
&= \sum_{t=0}^{\infty} \left\{q_t^0 - q_{t+1}^0 \left[\left(1-\tau_{t+1}^k\right) r_{t+1} + 1 - \delta\right]\right\} k_{t+1} - \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + \lim_{T\to\infty} q_T^0 k_{T+1}.
\end{aligned}$$

Substituting $u_c(t) = \lambda_t$ and $\frac{\lambda_t}{R_t} = \beta\lambda_{t+1}$ in the modified no-arbitrage condition leads to:

$$R_t - \left[(1-\tau_{t+1}^k) r_{t+1} + 1 - \delta_k\right] = \frac{u_k(t+1)}{u_c(t+1)}. \tag{4}$$

Multiplying both sides of (4) with q_{t+1}^0 and using the definition of the Arrow-Debreu price, we have

$$q_t^0 - q_{t+1}^0 \left[(1-\tau_{t+1}^k) r_{t+1} + 1 - \delta_k\right] = q_{t+1}^0 \frac{u_k(t+1)}{u_c(t+1)}. \tag{5}$$

Plugging (5) in the term (3) gives rise to

$$\sum_{t=0}^{\infty} q_t^0 x_t = \sum_{t=0}^{\infty} q_{t+1}^0 \frac{u_k(t+1)}{u_c(t+1)} k_{t+1} - \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + \lim_{T\to\infty} q_T^0 k_{T+1}. \tag{6}$$

Substituting (6) into equation (2) and imposing the following two transversality conditions

$$\lim_{T\to\infty} q_t^0 b_T = 0, \lim_{T\to\infty} q_T^0 k_{T+1} = 0,$$

we obtain the present-value budget constraint of the representative consumer

$$\sum_{t=0}^{\infty}\left[q_t^0 c_t + q_{t+1}^0 \frac{u_k(t+1)}{u_c(t+1)} k_{t+1}\right] = \sum_{t=0}^{\infty} q_t^0 (1-\tau_t^n) w_t n_t + \left[\left(1-\tau_0^k\right) r_0 + 1 - \delta\right] k_0 + b_0. \tag{7}$$

Substituting the price equations (1) and $\frac{u_l(t)}{u_c(t)} = (1-\tau_t^n) w_t$ into (7) and rearranging, we have the implementability condition:

$$\sum_{t=0}^{\infty} \beta^t [u_c(t) c_t - u_l(t) n_t + u_k(t) k_t] = u_c(0)\{[(1-\tau_0^k) r_0 + 1 - \delta_k] k_0 + b_0\} + u_k(0) k_0 \equiv \widetilde{A}_1 \tag{8}$$

Secondly, to solve the Ramsey problem, we form the Lagrangian

$$J = \sum_{t=0}^{\infty} \beta^t \{U(t) + \theta_t [F(k_t, n_t) - c_t - g_t - k_{t+1} + (1-\delta_k) k_t]\} - \Phi \widetilde{A}_1.$$

Note that

$$U(t) \equiv U(c_t, n_t, k_t, \Phi) \equiv u(c_t, 1-n_t, k_t) + \Phi[u_c(t) c_t - u_l(t) n_t + u_k(t) k_t],$$

where Φ is the Lagrangian multiplier w.r.t the IMC and $\{\theta_t\}_{t=0}^{\infty}$ is a sequence of Lagrangian multipliers. The first order conditions are

$$c_t: \quad U_c(t) = \theta_t, t \geq 1 \tag{9}$$

$$k_{t+1}: \quad \theta_t = \beta\{U_k(t+1) + \theta_{t+1}[F_k(t+1) + 1 - \delta_k]\}, t \geq 0 \tag{10}$$

$$n_t: \quad -U_n(t) = \theta_t F_n(t), t \geq 1 \tag{11}$$

where

$$\begin{aligned} U_c(t) &= u_c(t) + \Phi[u_{cc}(t) c_t + u_c(t) - u_{lc}(t) n_t + u_{kc}(t) k_t], \\ U_n(t) &= -u_l(t) + \Phi[-u_{cl}(t) c_t + u_{ll}(t) n_t - u_l(t) - u_{lk}(t) k_t], \\ U_k(t+1) &= u_k(t+1) + \Phi[u_{ck}(t+1) c_{t+1} - u_{lk}(t+1) n_{t+1} + u_{kk}(t+1) k_{t+1} + u_k(t+1)]. \end{aligned}$$

Finally, we examine the steady state of the economy. The steady state versions for equations (9)-(11) are

$$\theta = (1+\Phi) u_c + \Phi \underbrace{(u_{cc} c - u_{lc} n + u_{kc} k)}_{\equiv \eta_1}, \tag{12}$$

$$\theta[1-\beta(F_k+1-\delta_k)] = \beta[(1+\Phi)u_k + \Phi(\underbrace{u_{ck}c - u_{lk}n + u_{kk}k}_{\equiv\eta_2})], \tag{13}$$

$$\theta F_n = (1+\Phi)u_l + \Phi(\underbrace{u_{cl}c - u_{ll}n + u_{kl}k}_{\equiv\eta_3}). \tag{14}$$

From equations (12) and (14), we solve for $\frac{(1+\Phi)}{\theta}$ and $\frac{\Phi}{\theta}$ as follows:

$$\frac{(1+\Phi)}{\theta} = \frac{\eta_3 - F_n\eta_1}{u_c\eta_3 - u_l\eta_1}, \frac{\Phi}{\theta} = \frac{u_cF_n - u_l}{u_c\eta_3 - u_l\eta_1}. \tag{15}$$

From the consumption Euler equation, we know that

$$F_k + 1 - \delta_k = \frac{1}{\beta} - \frac{u_k}{u_c} + \tau^k F_k. \tag{16}$$

Dividing both sides of (13) by θ and plugging (15) and (16) into it, we obtain

$$\tau^k = \frac{1}{u_cF_k}\underbrace{\frac{(u_cF_n - u_l)}{(u_c\eta_3 - u_l\eta_1)}}_{=\frac{\Phi}{\theta}}(u_k\eta_1 - u_c\eta_2). \tag{17}$$

From equation(15), the term $(u_cF_n - u_l)/(u_c\eta_3 - u_l\eta_1) = \frac{\Phi}{\theta}$ is nonnegative, because the Lagrange multiplier Φ is nonnegative, while the insatiable utility function implies that θ is strictly positive. Notice that u_c and F_k are both strictly positive. Hence the sign of the limiting capital income tax is determined completely by the sign of the term $(u_k\eta_1 - u_c\eta_2)$. To examine the optimal labor income tax, we combine (12) with (14), rearrange the terms, and obtain

$$u_cF_n - u_l = \frac{\Phi}{1+\Phi}(\eta_3 - F_n\eta_1). \tag{18}$$

Substituting the marginal productivity condition of the firm into $\frac{u_l(t)}{u_c(t)} = (1-\tau_t^n)w_t$ gives us

$$u_cF_n - u_l = \tau^n u_cF_n. \tag{19}$$

Combining (18) with (19) leads to

$$\tau^n = \frac{1}{u_cF_n}\frac{\Phi}{1+\Phi}(\eta_3 - F_n\eta_1). \tag{20}$$

Since $u_c > 0$, $F_n > 0$ and the multiplier Φ is nonnegative, the limiting optimal labor income tax depends on the value of the term in the bracket, listed in the theorem.□

1.2 Proof of Proposition 4.2

Proof of Proposition 4.1. To solve the Ramsey problem, we construct the Lagrangian

$$L = \sum_{t=0}^{\infty}\beta^t \left[\alpha u^1(c_t^1, k_t^1) + (1-\alpha)\, u^2(c_t^2, 1-n_t^2)\right] + \widehat{\Phi}\left[\sum_{t=0}^{\infty}\beta^t[u_c^1(t)c_t^1 + u_k^1(t)k_t^1] - \widetilde{A}_2\right]$$
$$+\sum_{t=0}^{\infty}\beta^t\mu_t\left[u_l^2(t)n_t^2 - u_c^2\left(t\right)c_t^2\right] + \sum_{t=0}^{\infty}\beta^t\theta_t\left[F\left(k_t^1, n_t^2\right) - c_t^1 - c_t^2 - k_{t+1}^1 + (1-\delta)k_t^1 - g_t\right],$$

where $\widehat{\Phi}$, $\{\mu_t\}_{t=0}^{\infty}$ and $\{\theta_t\}_{t=0}^{\infty}$ are the Lagrange multipliers associated with the implementability condition, the optimality condition of the worker, and the resource constraint, respectively. The optimality conditions w.r.t c_t^1, k_{t+1}^1, c_t^2, and n_t^2 are:

$$\left(\alpha+\widehat{\Phi}\right)u_c^1(t) + \widehat{\Phi}\underbrace{\left[u_{cc}^1\left(t\right)c_t^1 + u_{kc}^1\left(t\right)k_t^1\right]}_{\equiv\varrho_1} = \theta_t, t\geq 1, \tag{21}$$

$$\beta\left\{\left(\alpha+\widehat{\Phi}\right)u_k^1(t+1) + \widehat{\Phi}\underbrace{\left[u_{kk}^1\left(t+1\right)k_{t+1}^1 + u_{ck}^1\left(t+1\right)c_{t+1}^1\right]}_{\equiv\varrho_2}\right\} = \theta_t - \beta\theta_{t+1}\left[F_k\left(t+1\right)+1-\delta\right], t\geq 0, \tag{22}$$

$$\left(1-\alpha-\mu_t\right)u_c^2(t) + \mu_t\underbrace{\left[u_{lc}^2\left(t\right)n_t^2 - u_{cc}^2\left(t\right)c_t^2\right]}_{\equiv\varrho_3} = \theta_t, t\geq 0, \tag{23}$$

$$\left(1-\alpha-\mu_t\right)u_l^2(t) + \mu_t\underbrace{\left[u_{ll}^2\left(t\right)n_t^2 - u_{cl}^2\left(t\right)c_t^2\right]}_{\equiv\varrho_4} = \theta_t F_n\left(t\right), t\geq 0. \tag{24}$$

Suppose that the economy converges to an interior steady state. Combining the steady state equations of the consumption Euler equation of the capitalist and (22) yields us

$$\tau^k = \frac{1}{F_k}\left[\frac{u_k^1}{u_c^1} - \frac{\alpha+\widehat{\Phi}}{\theta}u_k^1 - \frac{\widehat{\Phi}}{\theta}\varrho_2\right]. \tag{25}$$

Solving equation (21) for $\left(\alpha+\widehat{\Phi}\right)/\theta = \left(1-\widehat{\Phi}\varrho_1/\theta\right)/u_c^1$ and putting it into (25), we solve for

$$\tau^k = \frac{\widehat{\Phi}}{\theta}\frac{1}{u_c^1 F_k}\left(u_k^1\varrho_1 - u_c^1\varrho_2\right). \tag{26}$$

To search for the limiting labor income tax, we combine equations (23) and (24) to derive

$$\frac{u_l^2}{u_c^2} = \frac{(\theta F_n - \mu\varrho_3)}{(\theta-\mu\varrho_2)}. \tag{27}$$

Substituting (27) into the optimality condition of the representative worker, we obtain the formula for the limiting labor income tax

$$\tau^n = \frac{\varrho_4 - \varrho_3 F_n}{F_n}\frac{\mu}{\theta-\mu\varrho_3}. \tag{28}$$

The proof is completed. □